On p-valently convex and starlike functions of order α (New Extension of Historical Theorems for Univalent Function Theory)

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On p-valently convex and starlike functions of order α

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Abstract. The object of the present paper is to give the order of p-valently starlikeness for p-valently convex functions of order α in the open unit disk U.

1 Introduction

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n}$$

which are analytic in the open unit disk $U=\{z\in \mathbb{C}:|z|<1\}$.
For $0\leq\alpha<p$, if $f(z)\in A(p)$ satisfies the following condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in U),$$

then $f(z)$ is said to be p-valently starlike of order α, denoted by $S_{p}^{*}(\alpha)$ and if $f(z)\in A(p)$ satisfies the condition

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in U),$$

then $f(z)$ is said to be p-valently convex of order α, denoted by $C_{p}(\alpha)$.
Jack [3] obtained the following interesting theorem:
If $f(z)\in C_{1}(\alpha)$, then $f(z)\in S_{1}^{*}(\beta)$ where

$$\beta\geq\frac{2\alpha-1+\sqrt{9-4\alpha+4\alpha^{2}}}{4}.$$

The above estimate by Jack [1] is not sharp, and after this paper, MacGregor [4] and Wilken and Feng [7] settled this problem, their result is the following:
If $f(z)\in C_{1}(\alpha)$, then $f(z)\in S_{1}^{*}(\beta)$, where

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$$\beta = \begin{cases} (1-2\alpha)/2^{2-2\alpha}(1-2^{2\alpha-1}) & (\text{if } \alpha \neq 1/2) \\ 1/2\log 2 \quad (\text{if } \alpha = 1/2). \end{cases}$$

Very recently, Fukui, Saigo and Ikeda [2] obtained the following result:

If $f(z) \in C_p(\alpha)$, then $f(z) \in S_p^*(\beta)$, where $0 \leq \beta < p$ and
(a) for thc casc, $0 \leq \beta < p/2$, β must satisfics

$$\beta + \frac{\beta}{2(\beta - p)} \leq \alpha,$$

(b) for the case, $p/2 \leq \beta < p$,β must satisfies

$$\beta + \frac{2(\beta - p)}{\beta} \leq \alpha.$$

2 Main theorem

Theorem 1. *If $f(z) \in C_p(\alpha)$, then $f(z) \in S_p^*(\beta)$, where*

$$\beta = \frac{2p + 2\alpha - 1 - \sqrt{4p^2 + 4\alpha^2 + 1 - 8p\alpha - 4p - 4\alpha}}{4}.$$

Proof. Let us put

$$\frac{zf'(z)}{f(z)} = (p - \beta)\frac{1 + w(z)}{1 - w(z)} + \beta = \frac{(p - 2\beta)w(z) + p}{1 - w(z)},$$

where $0 \leq \beta < p, w(z)$ is analytic in U and $w(0) = 0$.
By the logarithmic differentiation, we have

$$1 + \frac{zf''(z)}{f'(z)} = (p - \beta)\frac{1 + w(z)}{1 - w(z)} + \beta + \frac{(p - 2\beta)zw'(z)}{(p - 2\beta)w(z) + p} + \frac{zw'(z)}{1 - w(z)}.$$

If there exists a point $z_0, |z_0| < 1$ such that

$$|w(z)| < 1 \qquad \text{for } |z| < |z_0|$$

and

$$|w(z_0)| = 1,$$

then from [3, Lemma 1], we have

$$z_0 w'(z_0) = kw(z_0), \qquad k \geq 1.$$

Therefore, it follows that

$$1+\mathrm{Re}\left\{\frac{z_0 f''(z_0)}{f'(z_0)}\right\} = \mathrm{Re}\left\{(p-\beta)\frac{1+w(z_0)}{1-w(z_0)}+\beta\right\}+\mathrm{Re}\left\{\frac{(p-2\beta)kw(z_0)}{(p-2\beta)w(z_0)+p}\right\}+\mathrm{Re}\left\{\frac{kw(z_0)}{1-w(z_0)}\right\}$$

$$= \beta+\frac{k}{2}-\frac{k}{2}\mathrm{Re}\left\{\frac{p-(p-2\beta)w(z_0)}{p+(p-2\beta)w(z_0)}\right\}-\frac{k}{2}+\frac{k}{2}\mathrm{Re}\left\{\frac{1+w(z_0)}{1-w(z_0)}\right\}$$

$$\leq \beta-\frac{\beta}{2(p-\beta)} = \frac{(2p-1)\beta-2\beta^2}{2(p-\beta)}.$$

Putting

$$\alpha = \frac{(2p-1)\beta-2\beta^2}{2(p-\beta},$$

then we have

$$\beta = \frac{2p+2\alpha-1-\sqrt{4p^2+4\alpha^2+1-8p\alpha-4p-4\alpha}}{4}.$$

This completes the proof of our theorem. □

Remark 1. In [1],[5] and [6], the following result was obtained:
If $f(z)\in C_p(0); 2\leq p$, then $f(z)\in S_p^*(0)$ and this result is sharp.
This paper can be the same situation as Jack's paper [3] contributed to MacGregor [4] and Wilken and Feng's theorem [7]. The author expect someone will obtain an exact result for this problem.

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